Indian Statistical Institute, Bangalore Centre.
Mid-Semester Exam : Topology I

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Answer for 50 marks. **Time Limit : 3 hours.**

Give necessary justifications and explanations for all your arguments. If you are citing results from the class, mention it clearly. Questions answered after 50 marks have been attempted will not be considered.

1. Let τ be a Hausdorff topology on a space X and τ_1 be a compact topology on X. Consider the following statements : (i) $\tau \subsetneq \tau_1$; (ii) $\tau_1 \subsetneq \tau_1$; (iii) $\tau_1 = \tau$. Explain with proofs or counterexamples, which of the above statements can hold ? Is it possible that both (i) and (ii) do not hold ? (8)

2. Let $X = \prod_{i=1}^{n} X_i$ be a product topological space. Show that X is sequentially compact as a topological space iff each X_i is sequentially compact. (10)

3. Let $X := \{f : \mathbb{N} \to \mathbb{R} : \sum_i (f(i))^2 < \infty\}$ be the space of square-summable functions. Show that $\|f\| = \sum_i (f(i))^2$ defines a norm on X that makes it a normed linear space under this norm. Which of the following four seperation properties (first countable, second countable, Lindelöf, seperable) does it satisfy ? Provide suitable justifications. (10)

4. Are continuous mappings of Cauchy sequences Cauchy ? What about uniformly continuous mappings ? Is the converse true i.e., if $\{f(x_n)\}$ is Cauchy whenever $\{x_n\}$ is Cauchy, is f uniformly continuous ? (7)

5. Let S_Ω be the minimal uncountable well-ordered subset. Which of the following properties are true for S_Ω. Provide suitable justifications. (10)

(a) First-countable.
(b) Countably compact i.e., every countable open cover has a finite sub-cover.
(c) Second-countable.
(d) Lindelöf
(e) Seperable.

6. Let $\beta_0(X)$ denote the cardinality of the set of connected components of X. Show that if $f : X \to Y$ is a continuous function then $\beta_0(Y) \leq \beta_0(X)$. What if f is an homeomorphism ? (7)

7. Let X be a compact metric space and also that $X \times X$ is also compact. Let $F : X \times X \to Z$ be a continuous function. Define $f_y(x) := F(x, y)$. Show that $\mathcal{A} := \{f_y : y \in X\}$ is an equicontinuous family of functions. Is this true for non-compact X ? (8)